

Representations of surface groups in the general linear group

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Abstract

We determine the number of connected components of the moduli space for representations of a surface group in the general linear group.

Key words: *Representations of fundamental groups of surfaces, Higgs
bundles, connected components of moduli spaces.*

MSC 2000: 14D20, 14F45, 14H60.

1 Introduction

Let X be a closed oriented surface of genus $g \geq 2$ and let \mathcal{M}_G^+ be the moduli space of semisimple representations of $\pi_1(X)$ in $\mathrm{GL}(n, \mathbb{R})$. In this paper we extend the results of Hitchin [7] on the number of connected components of $\mathcal{M}_{\mathrm{PSL}(n, \mathbb{R})}^+$ to the case of $\mathcal{M}_{\mathrm{GL}(n, \mathbb{R})}^+$ when $n \geq 3$. (The case of representations in $\mathrm{PSL}(2, \mathbb{R})$ was studied by Goldman [5] and the case of representations in $\mathrm{PGL}(2, \mathbb{R})$ and $\mathrm{GL}(2, \mathbb{R})$ was studied by Xia [11, 12].)

We adopt the Morse theoretic approach pioneered by Hitchin in [6, 7] and, indeed, most of our arguments follow [7] quite closely. However, at one point (Theorem 4.3 below), the application of a general result of [1] allows for a

significant simplification of the arguments (cf. [7, Lemma 9.6]). This provides a nice example of the power of the machinery introduced in that paper.

It is a pleasure to thank Nigel Hitchin for enlightening conversations on this topic. We also thank the referee for useful comments.

2 Representations and the moduli space

Let X be a closed oriented surface of genus g and let

$$\pi_1(X) = \langle a_1, b_1, \dots, a_g, b_g \mid \prod_{i=1}^g [a_i, b_i] = 1 \rangle$$

be its fundamental group. By a *representation* of $\pi_1(X)$ in $\mathrm{GL}(n, \mathbb{R})$ we understand a homomorphism $\rho: \pi_1(X) \rightarrow \mathrm{GL}(n, \mathbb{R})$. The set of all such homomorphisms, $\mathrm{Hom}(\pi_1(X), \mathrm{GL}(n, \mathbb{R}))$, can be naturally identified with the subset of $\mathrm{GL}(n, \mathbb{R})^k$ consisting of $2g$ -tuples $(A_1, B_1, \dots, A_g, B_g)$ satisfying the algebraic equation $\prod_{i=1}^g [A_i, B_i] = 1$. This shows that $\mathrm{Hom}(\pi_1(X), \mathrm{GL}(n, \mathbb{R}))$ is a real algebraic variety.

The group $\mathrm{GL}(n, \mathbb{R})$ acts on $\mathrm{Hom}(\pi_1(X), \mathrm{GL}(n, \mathbb{R}))$ by conjugation:

$$(g \cdot \rho)(\gamma) = g\rho(\gamma)g^{-1}$$

for $g \in \mathrm{GL}(n, \mathbb{R})$, $\rho \in \mathrm{Hom}(\pi_1(X), \mathrm{GL}(n, \mathbb{R}))$ and $\gamma \in \pi_1(X)$. If we restrict the action to the subspace $\mathrm{Hom}^+(\pi_1(X), \mathrm{GL}(n, \mathbb{R}))$ consisting of semi-simple representations, the orbit space is Hausdorff. Define the *moduli space* for representations of $\pi_1(X)$ in $\mathrm{GL}(n, \mathbb{R})$ to be the orbit space

$$\mathcal{M}_{\mathrm{GL}(n, \mathbb{R})}^+ = \mathrm{Hom}^+(\pi_1(X), \mathrm{GL}(n, \mathbb{R})) / \mathrm{GL}(n, \mathbb{R})$$

with the quotient topology.

Given a representation $\rho: \pi_1(X) \rightarrow \mathrm{GL}(n, \mathbb{R})$, there is an associated flat bundle on X , defined as $V_\rho = \tilde{X} \times_{\pi_1(X)} \mathbb{R}^n$, where $\tilde{X} \rightarrow X$ is the universal cover and $\pi_1(X)$ acts on \mathbb{R}^n via ρ . We then define invariants of ρ as the Stiefel–Whitney classes of V_ρ :

$$\begin{aligned} w_1(\rho) &= w_1(V_\rho) \in H^1(X, \mathbb{Z}/2) , \\ w_2(\rho) &= w_2(V_\rho) \in H^2(X, \mathbb{Z}/2) . \end{aligned}$$

For fixed $(w_1, w_2) \in H^1(X, \mathbb{Z}/2) \oplus H^2(X, \mathbb{Z}/2)$ we define a subspace

$$\mathcal{M}_{\mathrm{GL}(n, \mathbb{R})}^+(w_1, w_2) = \{\rho \mid w_i(\rho) = w_i, i = 1, 2\} \subseteq \mathcal{M}_{\mathrm{GL}(n, \mathbb{R})}^+ .$$

3 $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundles

Let G be a real reductive Lie group, let $H \subset G$ be a maximal compact subgroup and let $G_{\mathbb{C}}$ be the complexification of G . The complexification of H is denoted by $H_{\mathbb{C}} \subset G_{\mathbb{C}}$. At the Lie algebra level we have the Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. The restriction of the adjoint representation to H defines a representation on \mathfrak{m} called the *isotropy representation*; let

$$\iota: H_{\mathbb{C}} \longrightarrow \mathrm{GL}(\mathfrak{m}_{\mathbb{C}}) \tag{3.1}$$

be its complexification. There is a complex linear Lie algebra involution $\theta: \mathfrak{g}_{\mathbb{C}} \rightarrow \mathfrak{g}_{\mathbb{C}}$ which has $\mathfrak{h}_{\mathbb{C}}$ as its $+1$ -eigenspace and $\mathfrak{m}_{\mathbb{C}}$ as its -1 -eigenspace, giving the decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}}$.

In the case of interest to us, namely that of $G = \mathrm{GL}(n, \mathbb{R})$, the maximal compact subgroup is $H = \mathrm{O}(n)$ and its complexification is $H_{\mathbb{C}} = \mathrm{O}(n, \mathbb{C})$. In terms of a defining representation, we thus have a non-degenerate quadratic form Q on \mathbb{C}^n , and $\mathrm{O}(n, \mathbb{C})$ is the group of complex linear automorphisms of \mathbb{C}^n preserving the form Q . The corresponding decomposition at the Lie algebra level is $\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{o}(n, \mathbb{C}) \oplus \mathfrak{m}_{\mathbb{C}}$, where $\mathfrak{o}(n, \mathbb{C})$ and $\mathfrak{m}_{\mathbb{C}}$ are respectively the antisymmetric and symmetric endomorphisms of \mathbb{C}^n (with respect to Q).

Fix a complex structure on X . Since no confusion is likely to arise, we shall also call the corresponding Riemann surface X . We denote the canonical bundle of X by $K = T^*X^{1,0}$. One more piece of convenient notation is the following: for any Lie group G , if we have a principal G -bundle E on X and a G -space V , we denote the associated bundle with fibres V by $E(V) = E \times_G V$. For example, in the case of the representation (3.1) we obtain a bundle $E(\mathfrak{m}_{\mathbb{C}}) = E \times_{\iota} \mathfrak{m}_{\mathbb{C}}$ with fibres isomorphic to $\mathfrak{m}_{\mathbb{C}}$.

Definition 3.1. A G -Higgs bundle is a pair (E, Φ) , where $E \rightarrow X$ is a principal holomorphic $H_{\mathbb{C}}$ -bundle and the Higgs field Φ belongs to $H^0(E(\mathfrak{m}_{\mathbb{C}}) \otimes K)$.

A $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundle is thus a pair (E, Φ) , where E is a $\mathrm{O}(n, \mathbb{C})$ principal bundle and $\Phi \in H^0(E(\mathfrak{m}_{\mathbb{C}}) \otimes K)$. Letting $V = E \times_{\mathrm{O}(n, \mathbb{C})} \mathbb{C}^n$ be the associated vector bundle, we can describe this more concretely as a triple (V, Q, Φ) , where V is a holomorphic rank n vector bundle, $Q \in H^0(S^2 V^*)$ is a non-degenerate quadratic form and $\Phi \in H^0(\mathrm{End}(V) \otimes K)$ is symmetric with respect to Q . We denote by $q: V \rightarrow V^*$ the symmetric isomorphism associated to Q . In the decomposition

$$E(\mathfrak{g}_{\mathbb{C}}) = E(\mathfrak{h}_{\mathbb{C}}) \oplus E(\mathfrak{m}_{\mathbb{C}})$$

we have that $E(\mathfrak{g}_{\mathbb{C}})$ is just $\mathrm{End}(V)$, and the involution θ on $\mathrm{End}(V)$ defining the decomposition is

$$\theta: A \longmapsto -(q A q^{-1})^t. \tag{3.2}$$

Thus, under the isomorphism $q: V \rightarrow V^*$, we can identify the $+1$ -eigenbundle of θ ,

$$U^+ = E(\mathfrak{h}_{\mathbb{C}}) ,$$

as the bundle of antisymmetric endomorphisms of V and the -1 -eigenbundle of θ ,

$$U^- = E(\mathfrak{m}_{\mathbb{C}}) ,$$

as the bundle of symmetric endomorphisms of V .

The notion of isomorphism between G -Higgs bundles is the obvious one: (E, Φ) and (E', Φ') are isomorphic, if there is an isomorphism $g: E \xrightarrow{\cong} E'$ which takes Φ to Φ' under the induced isomorphism $E(\mathfrak{m}_{\mathbb{C}}) \xrightarrow{\cong} E'(\mathfrak{m}_{\mathbb{C}})$.

There is a stability condition for G -Higgs bundles, which generalizes the usual stability condition for Higgs vector bundles (see [6, 10]) and Ramanathan's stability condition for principal bundles [8]. It is a special case of the very general stability notion studied in [2]. We shall not need the detailed form of this stability condition here, but we mention that in the case of $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundles (V, Q, Φ) , it means that the usual slope condition is satisfied for Φ -invariant isotropic subbundles of V . We also note that there are corresponding notions of poly-stability and semi-stability.

One expects this stability condition to be appropriate for constructing moduli spaces of G -Higgs bundles. For this one should first fix the topological type of the principal $H_{\mathbb{C}}$ -bundle E . In the case of $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundles, this means fixing the first and second Stiefel–Whitney classes of any $\mathrm{O}(n)$ -bundle obtained by reduction of the structure group to the maximal compact subgroup $\mathrm{O}(n) \subseteq \mathrm{O}(n, \mathbb{C})$. In general no direct construction of the moduli spaces is currently available—in the case of present interest, we shall adopt the solution of [7], realizing the moduli space of semistable $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundles with fixed Stiefel–Whitney classes w_1 and w_2 as a subspace of the moduli space of ordinary Higgs (vector) bundles of rank n and degree 0. We denote the moduli space by $\mathcal{M}(w_1, w_2)$.

The G -Higgs bundle stability condition is equivalent to an existence criterion for solutions to certain gauge theoretic equations (Hitchin's equations) on (E, Φ) . This provides a bridge from poly-stable G -Higgs bundles to representations of $\pi_1(X)$, since solutions to the equations give rise to flat G -bundles. The existence of solutions was proved by Hitchin [6] and, more generally, Simpson [9, 10] in the case of Higgs bundles with complex structure group. However, in the present generality, this requires the results of [2].

The correspondence in the other direction is given by the theorem of Donaldson [4] and, more generally, Corlette [3]: given a flat semi-simple G -bundle, there is a preferred reduction of structure group to the maximal compact

$H \subset G$, a so-called harmonic metric. This gives rise to a solution to Hitchin's equations and thus to a (poly-)stable G -Higgs bundle (E, Φ) ; the $H_{\mathbb{C}}$ -bundle E of course being the complexification of the H -bundle obtained via the harmonic metric.

We thus have the following fundamental result, essentially due to Corlette, Donaldson, Hitchin and Simpson.

Theorem 3.2. *Let $w_1 \in H^1(X, \mathbb{Z}/2)$ and $w_2 \in H^2(X, \mathbb{Z}/2)$ be fixed. Then there is a homeomorphism between the moduli space of semistable G -Higgs bundles $\mathcal{M}(w_1, w_2)$ and the moduli space $\mathcal{M}_{\mathrm{GL}(n, \mathbb{R})}^+(w_1, w_2)$ of semisimple representations of the fundamental group of X in $\mathrm{GL}(n, \mathbb{R})$, with the given invariants.*

4 Morse theory on the Higgs bundle moduli space

Theorem 3.2 shows that counting the number of connected components of the moduli space of representations is the same thing as counting the number of connected components of the moduli space of $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundles. In order to do this we use the Morse theory approach to the study of the topology of Higgs bundle moduli introduced by Hitchin in [6, 7]. The L^2 -norm of Φ defines a positive function $f: \mathcal{M}(w_1, w_2) \rightarrow \mathbb{R}$, given by $f(E, \Phi) = \int_X \|\Phi\|^2$ (this definition uses the harmonic metric in the bundle E). When the Higgs bundle moduli space is smooth, this function is a perfect Bott–Morse function, giving a powerful tool for the study of the topology of the moduli space. But even when singularities are present, the fact that f is a proper map gives the following result on connected components.

Proposition 4.1. *Let $\mathcal{M}' \subseteq \mathcal{M}(w_1, w_2)$ be a subspace and let $\mathcal{N} \subseteq \mathcal{M}'$ be the subspace of \mathcal{M}' consisting of local minima of the restriction of f . If \mathcal{N} is connected, then so is \mathcal{M}' .*

Using this result for determining the connected components of $\mathcal{M}(w_1, w_2)$ obviously requires identifying the local minima of f on $\mathcal{M}(w_1, w_2)$. In order to do this, we use one of the main theorems proved in [1]. Before stating the result we need some preliminaries.

Any local minimum of f corresponds to a fixed point of the \mathbb{C}^* -action $(E, \Phi) \mapsto (E, \lambda\Phi)$. It is not hard to see that (E, Φ) represents a fixed point if and only if it is a so-called *complex variation of Hodge structure*. In the case of a $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundle (V, Q, Φ) , this means the following (cf. Hitchin [7] p. 466): the vector bundle V breaks up as a direct sum $V = F_{-m} \oplus \cdots \oplus F_m$ and the restriction Φ_i of the Higgs field Φ to F_i , gives maps $\Phi_i: F_k \rightarrow F_{k+1} \otimes K$.

Furthermore, the quadratic form Q gives an isomorphism $q: F_k \rightarrow F_{-k}^*$, and the remaining F_l are orthogonal to F_k under Q . Recalling that Φ is symmetric with respect to Q we thus have that

$$\Phi_{-k} = \Phi_{k-1}^t: F_{-k} \rightarrow F_{-k+1} \otimes K . \quad (4.1)$$

Define $U_{ij} = \text{Hom}(F_j, F_i)$ and $U_k = \bigoplus_{i-j=k} U_{ij}$. Then there is a corresponding decomposition of the Lie algebra bundle $E(\mathfrak{g}_{\mathbb{C}}) = \text{End}(V)$ as

$$\text{End}(V) = \bigoplus_{k=-2m}^{2m} U_k .$$

The restriction of the involution θ gives an isomorphism $\theta: U_{ij} \rightarrow U_{-j,-i}$ (cf. (3.2)). Hence θ restricts to $\theta: U_k \rightarrow U_k$. Letting $U_k = U_k^+ \oplus U_k^-$ be the corresponding eigenspace decomposition, we thus have $U^+ = \bigoplus U_k^+$ and $U^- = \bigoplus U_k^-$. The fact that Φ maps F_k to $F_{k+1} \otimes K$ means that $\Phi \in H^0(U_1^- \otimes K)$. Note that $\text{ad}(\Phi)$ interchanges U^+ and U^- , and therefore $\text{ad}(\Phi): U_k^\pm \rightarrow U_{k+1}^\mp \otimes K$. The result we need from [1] can now be stated as follows.

Theorem 4.2. *Let (V, Q, Φ) be a stable $\text{GL}(n, \mathbb{R})$ -Higgs bundle which represents a critical point of f . Then this critical point is a local minimum if and only if either $\Phi = 0$ or*

$$\text{ad}(\Phi): U_k^+ \rightarrow U_{k+1}^- \otimes K$$

is an isomorphism for all $k \geq 1$.

Proof. This follows from [1, Proposition 4.14] by an argument analogous to the proof of [1, Corollary 4.15] (cf. [1, Remark 4.16]). \square

Theorem 4.3. *Let the stable $\text{GL}(n, \mathbb{R})$ -Higgs bundle (V, Q, Φ) be a complex variation of Hodge structure. Assume that $n \geq 3$. Then (V, Q, Φ) represents a minimum of f if and only if one of the following two alternatives occurs:*

- (1) *The Higgs field Φ vanishes identically.*
- (2) *Each bundle F_i has rank 1 and, furthermore, the restriction $\Phi_i = \Phi|_{F_i}$ defines an isomorphism $\Phi_i: F_i \xrightarrow{\cong} F_{i+1} \otimes K$.*

Remark 4.4. In the case $n = 1$, it is easy to see that only minima of the type described in (1) of the Theorem occur. The case $n = 2$ is also special, since a third type of minima exists (see [7, Proposition 9.19]).

Proof of Theorem 4.3. It is clear from the definition of f as the L^2 -norm of Φ that $\Phi = 0$ implies that (E, Φ) is a local minimum. So assume from now on that $\Phi \neq 0$ and represents a local minimum of f .

Consider first $U_{m,-m} = \text{Hom}(F_{-m}, F_m) = U_{2m}$. Then we have the isomorphism $\theta: U_{m,-m} \xrightarrow{\cong} U_{m,-m}$ and the $+1$ -eigenspace $U_{m,-m}^+$ is the space of antisymmetric maps $F_{-m} \rightarrow F_m$ under the duality $q: F_m \xrightarrow{\cong} F_{-m}^*$. Theorem 4.2 says that we have an isomorphism $\text{ad}(\Phi): U_{2m}^+ \rightarrow U_{2m+1}^- \otimes K$. Since the latter space is 0, so is the former. Hence we conclude that there are no antisymmetric maps $F_{-m} \rightarrow F_m$. This is only possible if $\text{rk}(F_m) = \text{rk}(F_{-m}) = 1$.

Next we shall prove that the remaining F_i are line bundles and that $\Phi_i: F_i \rightarrow F_{i+1} \otimes K$ are isomorphisms. Note that we only need to do this for $i \geq 0$ (cf. (4.1)). We proceed by induction, taking as induction hypothesis that F_{m-l} has rank 1 for $0 \leq l \leq k$, and show for $k \leq m-1$ that F_{m-k-1} has rank 1 and that we have an isomorphism $\Phi_{m-k-1}: F_{m-k-1} \xrightarrow{\cong} F_{m-k} \otimes K$.

Consider $U_{m-(k+1),-m}$ and $U_{m,k+1-m}$. These spaces are transformed into each other by θ and, if we organize the U_{ij} into a matrix, they are located at opposite extremes of the diagonal whose elements make up $U_{2m-(k+1)}$. Thus elements of the form $(a, \theta(a))$ in the direct sum $U_{m-(k+1),-m} \oplus U_{m,k+1-m}$ belong to $U_{2m-(k+1)}^+$. By Theorem 4.2 we know that the restriction of $\text{ad}(\Phi)$ to the subspace of $U_{2m-(k+1)}^+$ consisting of such pairs $(a, \theta(a))$ is injective when $k \leq 2m-2$. This condition is satisfied, since $k \leq m-1$ (this is where we need $n \geq 3$, in order to have $m-1 \geq 0$). We have¹

$$\text{ad}(\Phi)(a, \theta(a)) = \Phi_{m-(k+1)} \circ a - \theta(a) \circ \Phi_{k-m} . \quad (4.2)$$

If $k=0$ these two summands both lie in $U_{m,-m}$. We calculate the second one, using that $\Phi_{-m} = \theta(\Phi_{m-1}) = (q\Phi_{m-1}q^{-1})^t$:

$$\theta(a) \circ \Phi_{-m} = -(qaq^{-1})^t \circ (q\Phi_{m-1}q^{-1})^t = -(q\Phi_{m-1}aq^{-1})^t = \theta(a\Phi_{m-1}) .$$

But we have already proved that $U_{m,-m}$ is a line bundle on which θ is $+1$. Hence (4.2) shows that when $k=0$

$$\text{ad}(\Phi)(a, \theta(a)) = 2\Phi_{m-1} \circ a .$$

Since $\text{ad}(\Phi)$ is injective, this proves that $\Phi_{m-1}: F_{m-1} \rightarrow F_m \otimes K$ injects. But F_m is a line bundle, so F_{m-1} must also be a line bundle and Φ_{m-1} an isomorphism.

¹In this formula $\theta(a)$ should be twisted by the identity on K . Since no confusion can arise, we shall discard such twisting from the notation.

A similar argument works when $k \geq 1$; however this case is easier, because the summands in (4.2) lie in different $U_{i,j}$ and hence we can appeal to injectivity of $(a, \theta(a)) \mapsto \Phi_{m-k+1} \circ a$. \square

Let (V, Q, Φ) be a local minimum of f of the kind described in (2) of Theorem 4.3. Using $F_k \cong F_{-k}^*$ we see that $n = 2m + 1$ and that m is half integer when n is even, while m is integer when n is odd. Hence

$$\begin{aligned} V &= F_{-m} \oplus \cdots \oplus F_{-1/2} \oplus F_{1/2} \oplus \cdots \oplus F_m && \text{if } n \text{ is even,} \\ V &= F_{-m} \oplus \cdots \oplus F_0 \oplus \cdots \oplus F_m && \text{if } n \text{ is odd.} \end{aligned}$$

Thus Theorem 4.3 leads to the following more precise characterization of the (V, Q, Φ) with $\Phi \neq 0$ representing a local minimum of f .

Proposition 4.5. *Let (V, Q, Φ) be a local minimum of f of the kind described in (2) of Theorem 4.3. Then the following holds.*

(1) *If n is even, then $F_{-1/2}^2 = K$ and the remaining F_i are uniquely determined by the choice of this square root of K as $F_{-1/2+k} \cong F_{-1/2} \otimes K^{-k}$.*

(2) *If n is odd, then $F_0^2 = \mathcal{O}$ and $F_k \cong F_0 \otimes K^{-k}$ for $k \neq 0$.*

(3) *In both cases, (V, Q, Φ) is isomorphic to a $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundle, where*

$$q = \begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ & & & \cdots & 0 \\ \vdots & & & & \\ & & 1 & & \vdots \\ 0 & \cdots & & & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix} \quad \text{and} \quad \Phi = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

with respect to the decomposition $V = F_{-m} \oplus \cdots \oplus F_m$. (Here each F_i is a line bundle and $n = 2m + 1$. In the notation for q : $V \xrightarrow{\cong} V^$ we use $F_i \cong F_{-i}^*$ and in the notation for Φ we use 1 for the canonical identification $F_0 \otimes K^{-k} \cong F_0 \otimes K^{-(k+1)} \otimes K$.)*

Proof. (1) and (2) are clear, using the isomorphism $\Phi_i: F_i \xrightarrow{\cong} F_{i+1} \otimes K$. For (3) we note that (V, Q, Φ) is of the form given, except that the 1's appearing in Φ are arbitrary non-zero complex scalars $\lambda_1, \dots, \lambda_{n-1}$. It is an easy exercise to show that such a (V, Q, Φ) is isomorphic to a $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundle of the kind given. \square

Now we calculate the invariants (w_1, w_2) of the (V, Q, Φ) described in the preceding Proposition. To be precise, we need to calculate the Stiefel–Whitney classes of the real bundle obtained by a reduction of structure group in (V, Q) from $\mathrm{O}(n, \mathbb{C})$ to $\mathrm{O}(n)$. We shall denote these classes by $w_i(V, Q)$.

Proposition 4.6. *Let (V, Q, Φ) be a local minimum of f of the kind described in (2) of Theorem 4.3. If $n = 2q$ is even, then*

$$\begin{aligned} w_1(V, Q) &= 0, \\ w_2(V, Q) &= (g-1)q^2 \pmod{2}, \end{aligned}$$

If $n = 2q+1$ is odd, then

$$\begin{aligned} w_1(V, Q) &= w_1(F_0, Q|_{F_0}), \\ w_2(V, Q) &= 0. \end{aligned}$$

Proof. In the case of the first Stiefel-Whitney class w_1 , consider the determinant bundle $\Lambda^n V$ in the Jacobian of X . Since $V \cong V^*$, we have $(\Lambda^n V)^2 = \mathcal{O}$. It is then easy to see that $\Lambda^n V$ corresponds to $w_1(V)$ under the identification of the 2-torsion points in the Jacobian with $H^1(X, \mathbb{Z}/2)$. From this the calculation of $w_1(V, Q)$ is immediate.

For the calculation of w_2 we can get a reduction of structure group from $O(n, \mathbb{C})$ to $O(n)$ in the bundle (V, Q) as follows: choose a Hermitian metric on V and compose the corresponding isomorphism $V^* \rightarrow \bar{V}$ with the isomorphism $q: V \rightarrow V^*$ to obtain the real structure on V . We shall take a hermitian metric under which the F_i are orthogonal.

When $n = 2q$ is even, we see that the underlying real bundle is

$$F_{1/2} \oplus \cdots \oplus F_m = F_{-1/2} \otimes (K^{-1} \oplus \cdots \oplus K^{-q}).$$

Recalling that $F_{-1/2}^2 = K$, this gives the formula stated for w_2 .

When $n = 2q+1$ is odd, we see that the underlying real bundle is

$$F_{0,\mathbb{R}} \oplus F_1 \oplus \cdots \oplus F_m,$$

where $F_{0,\mathbb{R}}$ is the real bundle underlying F_0 in the real structure defined by $q: F_0 \rightarrow F_0^*$ and the hermitian metric. We have that $w_1(F_i) = 0$ and $w_2(F_i) = c_1(F_i) = 0 \pmod{2}$, since $F_i = F_0 \otimes K^{-i}$ has even degree. Hence $w_i(V, Q) = w_i(F_{0,\mathbb{R}})$. To complete the proof we only need to note that $w_2(F_{0,\mathbb{R}}) = c_1(F_0) \pmod{2} = 0$. \square

One of the main results of [7] is the construction of a “Teichmüller component” of the moduli space of representations of $\pi_1(X)$ in any split real form of a complex simple Lie group. The construction in the case of representations in $SL(n, \mathbb{R})$ is quite explicit (see [7, §3]): one keeps the same underlying bundle and adds certain extra entries to the matrix of the Higgs field, thus parametrizing the component by spaces of sections of powers of K . The construction carries over to the case of $GL(n, \mathbb{R})$ -Higgs bundles with only one small modification: in [7, (3.2)], substitute the zero in the bottom row of the matrix with $\alpha_0 \in H^0(X, K)$. We thus have the following result.

Proposition 4.7. *Let (V, Q, Φ) be a local minimum of f of the kind described in (2) of Theorem 4.3 with invariants (w_1, w_2) . Then there is a connected component of $\mathcal{M}(w_1, w_2)$ containing (V, Q, Φ) and isomorphic to a vector space. Any (V, Q, Φ) in this component has $\Phi \neq 0$.*

5 Connected components of the moduli space

In this section we finally determine the connected components of $\mathcal{M}_{\mathrm{GL}(n, \mathbb{R})}^+$. In order to deal with possible non-stable minima we need the following result. The proof given at the beginning of §10 of [7] for $G = \mathrm{SL}(n, \mathbb{R})$ also works for $G = \mathrm{GL}(n, \mathbb{R})$.

Lemma 5.1. *Any (V, Q, Φ) representing a local minimum of f and which has $\Phi \neq 0$ is a stable $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundle.*

We can now state and prove our main Theorem.

Theorem 5.2. *Let $\mathcal{M}_{\mathrm{GL}(n, \mathbb{R})}^+$ be the moduli space of semi-simple representations in $\mathrm{GL}(n, \mathbb{R})$ of the fundamental group of a closed oriented surface of genus $g \geq 2$. Assume that $n \geq 3$. For $(w_1, w_2) \in H^1(X, \mathbb{Z}/2) \oplus H^2(X, \mathbb{Z}/2)$ let $\mathcal{M}_{\mathrm{GL}(n, \mathbb{R})}^+(w_1, w_2)$ be the subspace of representations with invariants (w_1, w_2) .*

(1) *If $n = 2q$ is even, then $\mathcal{M}_{\mathrm{GL}(n, \mathbb{R})}^+$ has $3 \cdot 2^{2g}$ connected components. More precisely:*

- (i) *If $w_1 \neq 0$, then $\mathcal{M}_{\mathrm{GL}(n, \mathbb{R})}^+(w_1, w_2)$ is connected.*
- (ii) *If $w_1 = 0$ and $w_2 \neq (g-1)q^2 \pmod{2}$, then $\mathcal{M}_{\mathrm{GL}(n, \mathbb{R})}^+(w_1, w_2)$ is connected.*
- (iii) *If $w_1 = 0$ and $w_2 = (g-1)q^2 \pmod{2}$, then $\mathcal{M}_{\mathrm{GL}(n, \mathbb{R})}^+(w_1, w_2)$ has $2^{2g} + 1$ connected components.*

(2) *If $n = 2q + 1$ is odd, then $\mathcal{M}_{\mathrm{GL}(n, \mathbb{R})}^+$ has $3 \cdot 2^{2g}$ connected components. More precisely:*

- (i) *If $w_2 \neq 0$, then $\mathcal{M}_{\mathrm{GL}(n, \mathbb{R})}^+(w_1, w_2)$ is connected.*
- (ii) *If $w_2 = 0$, then $\mathcal{M}_{\mathrm{GL}(n, \mathbb{R})}^+(w_1, w_2)$ has 2 connected components.*

Proof. By Theorem 3.2, $\mathcal{M}_{\mathrm{GL}(n, \mathbb{R})}^+(w_1, w_2)$ has the same number of connected components as $\mathcal{M}(w_1, w_2)$.

Any connected component of this space must contain a minimum of the non-negative proper map f . Thus, combining Theorem 4.3 and Lemma 5.1,

each component contains either a $GL(n, \mathbb{R})$ -Higgs bundle (V, Q, Φ) with $\Phi = 0$, or one with $\Phi \neq 0$ (i.e. of the form given in (2) of Theorem 4.3). Proposition 4.7 shows that for each isomorphism class of a minimum (V, Q, Φ) of f with $\Phi \neq 0$, there is a connected component of the corresponding $\mathcal{M}(w_1, w_2)$. Since $\Phi \neq 0$ for all (V, Q, Φ) in this component, it is disjoint from any component with a minimum with $\Phi = 0$.

For each value of (w_1, w_2) we let $\mathcal{M}_0(w_1, w_2)$ be the space $\mathcal{M}(w_1, w_2)$ with any components with minima with $\Phi \neq 0$ removed. The space $\mathcal{M}(w_1, w_2)$ contains the moduli space of semistable principal $O(n, \mathbb{C})$ -bundles with the same invariants (w_1, w_2) , included as $(V, Q) \mapsto (V, Q, 0)$. From Theorem 4.3 we conclude that this is exactly the space of local minima of f on $\mathcal{M}_0(w_1, w_2)$. But from Ramanathan [8, Proposition 4.2], we know that the moduli space of semistable principal $O(n, \mathbb{C})$ -bundles is connected. Hence Proposition 4.1 shows that $\mathcal{M}_0(w_1, w_2)$ is a connected component.

In conclusion we then have one connected component $\mathcal{M}_0(w_1, w_2)$ for each value of (w_1, w_2) and an “extra” connected component for each isomorphism class of a minimum (V, Q, Φ) of f with $\Phi \neq 0$. With the aid of Propositions 4.5 and 4.6 this leads to the statement of the Theorem. \square

Remark 5.3. When n is even, our count differs somewhat from the count for $G = PSL(n, \mathbb{R})$ in [7]:

- (1) The existence of a different connected component for each choice of a square root $F_{-1/2}$ of K (cf. (1) of Proposition 4.5), giving rise to the 2^{2g} components with $\Phi \neq 0$ in (iii) of (1) of Theorem 5.2 does not occur in [7]. This is because the Higgs bundles in question are projectively equivalent.
- (2) It was observed in [7, p. 473] that, when n is even, the components corresponding to minima with $\Phi \neq 0$ appear twice in the moduli space of $PSL(n, \mathbb{R})$ -representations. This happens because in that paper $PSL(n, \mathbb{R})$ -representations are analyzed, in the first place, up to $PGL(n, \mathbb{R})$ -equivalence. Since we are dealing here with $GL(n, \mathbb{R})$ -representations up to $GL(n, \mathbb{R})$ -equivalence, this phenomenon does not occur in our case.

Acknowledgments

The authors are members of VBAC (Vector Bundles on Algebraic Curves), which is partially supported by EAGER (EC FP5 Contract no. HPRN-CT-2000-00099) and by EDGE (EC FP5 Contract no. HPRN-CT-2000-00101). The second and third authors were partially supported by the Portugal/Spain bilateral Programme Acciones Integradas, grant nos. HP2000-0015 and AI-01/24. The first author was partially supported by the National Science Foundation under grant DMS-0072073. The second author was partially supported

by the Ministerio de Ciencia y Tecnología (Spain) under grant BFM2000-0024. The third author was partially supported by the Fundação para a Ciência e a Tecnologia (Portugal) through the Centro de Matemática da Universidade do Porto.

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